SCISSOR CONGRUENCE*

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ABSTRACT

It is shown that certain simple figures can not be cut by scissors into pieces that can be reassembled to form certain other simple figures.

Bolyai has shown that every convex polygon of unit area can be cut by a finite number of line segements into a finite number of pieces which can then be rearranged to form the unit square (see, for example, [1]). We show that the only convex bodies that can thus be rearranged are polygons even if the scissors are permitted to cut along arbitrary Jordan curves. Similarly, a circle of radius two cannot be cut by Jordan scissors into pieces that can be reassembled to form four circles of radius 1. Variants of the problems studied here already occur in Euclid and have been studied up to recent times. (For reference, for example to work of Banach and Tarski, see [4] and [5]). This paper, though self-contained, is, in a sense, a sequel to that of Rodrigues [2].

An easily stated result is

PROPOSITION 1. *Suppose that E and E' are strictly convex planar bodies. Then E and E' are scissor-congruent if and only if they have the same area and their respective boundaries B and B' are scissor-congruent.*

Definition of scissor congruence. A topological disc D is the image of the unit disc under a homeomorphism of the plane onto itself, or equivalently, is the interior and boundary of a simple, closed Jordan curve. Let $\operatorname{int} D$ = interior of D, bd D = boundary of D, $\operatorname{ext} D$ = exterior of D = complement of D, and let Φ be the empty set.

Throughout the first 15 lemmas these notations are used.

(a) $D_1, \cdots, D_n, D_{-1}, \cdots, D_{-n}$ are $2n$ topological discs (included in a fixed 2-dimensional Euclidean plane π).

(b) $T_1, \cdots, T_n$ are n rigid motions (of π onto itself).

(c) E_+ is the set-theoretic union of D_i for $i > 0$;
E_- is the union of D_i for $i < 0$.

(d) K_+ is the boundary of E_+;
K_- is the boundary of E_-.

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(e) $E = E_+ \cup E_-$; $K = K_+ \cup K_-$.

(f) J_i is the boundary of D_i for all i.

Throughout the first 15 lemmas these four assumptions are implicit.

(i) The image of D_i under T_i is D_{-i}, for $i = 1, \cdots, n$.

(ii) int $D_i \cap$ int $D_j = \Phi$ for $i > j > 0$;
int $D_i \cap$ int $D_j = \Phi$ for $i < j < 0$.

The joint assumptions (i) and (ii) are abbreviated to: E_+ is *scissor-congruent* to E_-.

The next assumption is automatic in the most interesting case in which E_+ and E_- are themselves topological discs (as in Proposition 1) or even finite unions of disjoint topological discs (as in Theorem 1).

(iii) $J_i \cap J_j \cap K_+$ is a finite set for $i > j > 0$;
$J_i \cap J_j \cap K_-$ is a finite set for $i < j < 0$.

The following assumption is of no mathematical importance and is made mainly to simplify some ensuing notation.

(iv) $E_+ \cap E_- = \Phi$.

An *arc* A is the image of a connected subset A' of the circumference C of a circle under a homeomorphism (of the plane onto itself) except that the empty set, C, and one-point sets are not considered as arcs. A is *open* if A' is open in C.

If K_+ is the disjoint union of a finite number of arcs $A_1, \cdots, A_r$ and a finite number of points and K_- is the disjoint union of a finite number of arcs $A_{-1}, \cdots, A_{-r}$ and a finite number of points and for each i, $1 \leqq i \leqq r$, A_{-i} is the image of A_i under a rigid motion R_i, then K_+ is *scissor-congruent* to K_-.

Circles and squares are not rectifiably scissor-congruent. A proof that a circular disc S cannot be partitioned into pieces with *rectifiable* boundaries which can be rearranged so as to form a square S' will be given here. Though it seems impossible to modify this proof so as to apply to the case of nonrectifiable boundaries, we present it now because it is so simple and the underlying idea pervades the more complicated proof of Theorem 1. It may however be skipped without logical loss for it is superseded by the general argument below.

If A is an arc on the boundary of a disc D, say that A is *convex relative* to D if the line segment joining every pair of points of A is a subset of D; say that A is *concave relative to* D if the line segment joining every pair of points of A is disjoint from the interior of D. For each disc D, and each point x on the boundary of D, let $f_D(x) = +1$ [respectively -1] if there is an arc A containing x in its interior that is congruent to a subarc of the circular boundary of S and which is convex [concave] relative to D; let $f_D(x) = 0$ otherwise. If the boundary of D is rectifiable, then f_D may be integrated with respect to the arc length measure determined by the boundary, obtaining thereby a number $\mu(D)$. More generally, these definitions are applicable if D is any set such as E_+ above. The measure μ is easily seen to be invariant under rigid motions — $\mu(D) = \mu(M(D))$ for all isometries M — and to be additive — $\mu(D_1 \cup D_2) = \mu(D_1) + \mu(D_2)$ whenever the intersection of D_1 and D_2 consists of at most a finite number of arcs. These two properties of μ imply that $\mu(D_1) = \mu(D_2)$ whenever D_1 and D_2 are scissor-congruent. Since $\mu(S) = 2\pi R$ where R is the radius of S, and $\mu(S') = 0$, S and S' are not scissor-congruent if only rectifiable cuts are admissible. This argument obviously applies to cubes and balls in any finite dimensional Euclidean space.

The main proof. For any x, let $N(x)$ mean neighborhood of x. The following topological properties are obvious for the unit disc and consequently hold for any topological disc D. For any x in the boundary of D and any $N(x)$:

(a) $N(x) \cap \operatorname{int} D \neq \Phi$;

(b) $N(x) \cap \operatorname{ext} D \neq \Phi$;

(c) $(N(x) - \{x\}) \cap bd\, D \neq \Phi$;

(d) There is an $N'(x) \subset N(x)$ such that $N'(x) \cap \operatorname{int} D$ and $N'(x) \cap \operatorname{ext} D$ are connected.

Note the following. If $A_1, A_2, \cdots$ is a sequence of disjoint arcs $\subset J$, where $J(\subset \text{plane})$ is homeomorphic to the unit circle C, then diameter $A_n \to 0$ as $n \to \infty$. Since the homeomorphism between C and J is uniformly continuous it suffices to show this for $J = C$. But if $J = C$, the sum of the diameters of the A_n is less than or equal to the sum of the arc lengths of the A_n which, in turn, is at most 2π. This implies that *no Jordan arc contains infinitely many, pairwise-disjoint, congruent subarcs*. Notice also that K is the boundary of E.

LEMMA 1. *For all* i, $J_i - \bigcup_{j \neq i} J_j \subset K$.

Proof. Let $x \in J_i - \bigcup_{j \neq i} J_j$. Since $\bigcup_{j \neq i} J_j$ is closed, for every $\varepsilon > 0$ there is a disc $D(x)$ of radius less than ε centered at x, such that $D(x) \cap J_j = \Phi$, $j \neq i$, so

$$D(x) = (D(x) \cap \operatorname{int} D_j) \cup (D(x) \cap \operatorname{ext} D_j).$$

Since $D(x)$ is connected, either $D(x) \subset \operatorname{int} D_j$ or $D(x) \subset \operatorname{ext} D_j$. Of course, $D(x) \cap \operatorname{int} D_i \neq \Phi$ and $D(x) \cap \operatorname{ext} D_i \neq \Phi$. Recall that $\operatorname{int} D_i \cap \operatorname{int} D_j = \Phi$ for $j \neq i$; consequently $D(x) \subset \operatorname{ext} D_j$ for $j \neq i$, and hence $D(x) \cap \operatorname{ext} E \neq \Phi$; so $x \in K$.

DEFINITION.

$$V = \bigcup_{\substack{i \neq j \neq k \\ i \neq k}} (J_i \cap J_j \cap J_k) \cup \bigcup_{i \neq j} (J_i \cap J_j \cap K).$$

As is not difficult to verify, (ii) together with (iii) imply:

(v) V is a finite set.

(At the slight cost of a not very important redundancy, (v) can be assumed instead of verified.)

LEMMA 2. *Let A be an arc, $A \subset (J_i - V)$. Then $A \subset K$ or $A \subset J_j$ for some $j \neq i$.*

Proof. By Lemma 1, $A = (\bigcup_{j \neq i} A \cap J_j) \cup (A \cap K)$, so A is a union of sets closed in A whose pairwise intersections are in $A \cap V = \Phi$. Since A is connected, Lemma 2 is obvious.

DEFINITIONS. For $i > 0$, $T_{-i} = T_i^{-1}$. It is convenient to introduce two transformations T and R defined for certain ordered couples (x, i). For $x \in J_i$, let $T(x,i) = (T_i x, -i)$. For $x \in J_i \cap J_j - V$, with $i \neq j$, let $R(x,i) = (x,j)$; since $x \notin V$, j is unique, so R is well-defined.

Denote the identity map of a set X onto itself by 1_X.

LEMMA 3. (a) *$\mathscr{D}(R)$, that is, the domain of R, is the set of all (x,i) such that $x \in J_i$ and $x \notin K \cup V$.*

(b) *$\mathscr{R}(R)$, that is, the range of R, is the same as $\mathscr{D}(R)$, and $R^2 = 1_{\mathscr{D}(R)}$.*

(c) *$\mathscr{R}(T) = \mathscr{D}(T)$, and $T^2 = 1_{\mathscr{D}(T)}$.*

(d) $\mathscr{R}((TR)^k T) = \mathscr{D}((TR)^k T)$, and $((TR)^k T)^2 = 1_{\mathscr{D}((TR)^k T)}$, *for each $k \geqq 0$.*

Proof. (a) follows from Lemma 1; (b) and (c) are immediate; and (d) follows from (b) and (c), using induction on k.

LEMMA 4. *Assume:*

(a) *x and y are in K,*

(b) $(x,i) \in \mathscr{D}((TR)^k T)$,

and

(c) $(y,j) \in \mathscr{D}((TR)^h T)$.

Then $(TR)^k T(x,i) = (TR)^h T(y,j)$ only if $k = h$ and $(x,i) = (y,j)$.

Proof. Suppose $k \leqq h$. Then $(TR)^h T = (TR)^k T(RT)^{h-k}$. By Lemma 3(d), $(x,i) = (RT)^{h-k}(y,j)$. If $k = h$, then $(x,i) = (y,j)$. If $k < h$, then $(x,i) \in \mathscr{R}(R) = \mathscr{D}(R)$ by Lemma 3(c), so $x \notin K$ by Lemma 3(a), which is a contradiction.

DEFINITION. $W = \{x : x \in K$ and there exist i, k, v, i' such that $v \in V$, $(x,i) \in \mathscr{D}((TR)^k T)$ and $(TR)^k T(x,i) = (v,i')\}$.

LEMMA 5. *W is finite.*

Proof. For every $x \in W$, the set of all (v,i') such that for some (i,k), (i,k,v,i') satisfies the conditions of the definition of W is nonempty. By Lemma 4, distinct x's correspond to disjoint sets. Hence, the cardinality of W does not exceed that of $\{(v,j) : v \in V\}$, which is finite, since V is finite.

LEMMA 6. *$K-(V \cup W)$ is a finite union of disjoint open arcs, each included in some J_i, and this decomposition is unique.*

Proof. Clearly, $K \subset \bigcup_i J_i$, so $K-(V \cup W)=\bigcup_i [K \cap (J_i-(V \cup W))]$. Since $V \cup W$ is finite, $J_i-(V \cup W)$ is uniquely expressible as a finite union of disjoint open arcs (its connected components). By Lemma 2, every open arc in the decomposition of $J_i-(V \cup W)$ is either included in K or is disjoint from K. Therefore, $K \cap (J_i-(V \cup W))$ has such a unique decomposition, and since the $K \cap (J_i-(V \cup W)) = (K-(V \cup W)) \cap J_i$ are disjoint sets closed in their union $K-(V \cup W)$, the result follows.

DEFINITION. $p(x)=x'$ if $x,x' \in K-(V \cup W)$ and there exist i,i',k such that

(1) $$(x,i) \in \mathscr{D}((TR)^kT) \text{ and } (TR)^kT(x,i)=(x',i').$$

Here is a proof that p is well defined and $p^2 = 1_{\mathscr{D}(p)}$.

If (x,i,k,x',i') and (x,j,h,y',j') satisfy (1), then so do (x',i',k,x,i) and (y',j',h,x,j) by Lemma 3(d). Since both (x,i) and (x,j) are in $\mathscr{D}(T)$, $x \in J_i \cap J_j$. But $x \in K-V$, so $i=j$.

By Lemma 4, $x'=y'$, so p is well-defined, and $p^2 = 1_{\mathscr{D}(p)}$.

DEFINITION. Let $\mathscr{A}$ be the finite set of open arcs described in Lemma 6.

LEMMA 7. *If $A \in \mathscr{A}$, then $A \subset \mathscr{D}(p)$, and, therefore, $\mathscr{D}(p)=K-(V \cup W)$. Moreover, there exist $i_0,i_1,\cdots,i_k$ such that for $0 \leqq r \leqq k$:*

(a) $-i_r \neq i_{r+1}$;

(b) $A \subset J_{i_0}$;

and, letting A_r denote $T_{i_r}\cdots T_{i_0}A$,

(c) $A_r \subset J_{-i_r} \cap J_{i_{r+1}}$;

(d) $p(A)=A_k \subset J_{-i_k} \cap (K-(V \cup W))$.

Proof. Fix $A \in \mathscr{A}$. Then A is a subset of some unique J_i, say $i=i_0$.

Let (A,i) be the set of all (x,i) for which $x \in A$. Then $(A,i_0) \subset \mathscr{D}(T)$, and $T(A,i_0)=(A_0,-i_0)$, where $A_0=T_{i_0}A$ is congruent to A. The immediate goal is to show that if $(r,A_r,-i_r)$ satisfies

(2) $(A,i_0) \subset \mathscr{D}((TR)^rT)$ and $(TR)^rT(A,i_0)=(A_r,-i_r)$, then either

$$A_r \subset J_{-i_r} \cap (K-(V \cup W),$$

or

$$(r+1, A_{r+1}, -i_{r+1}) \text{ satisfies (2),}$$

where $A_r \subset J_{-i_r} \cap J_{i_{r+1}}$, $-i_r \neq i_{r+1}$, and $A_{r+1}=T_{i_{r+1}}A_r$.

Suppose $(r,A_r,-i_r)$ satisfies (2). Since $A \cap W$ is empty, $A_r \cap V$ certainly is empty; since A_r is congruent to A, A_r is an arc; and $A_r \subset J_{-i_r}$. By Lemma 2, $A_r \subset K$ or $A_r \subset J_{i_{r+1}}$ for some $i_{r+1} \neq -i_r$. In the latter case, $(A_r,-i_r) \subset \mathscr{D}(R) = \mathscr{D}(TR)$, so $(A,i_0) \subset \mathscr{D}((TR)^{r+1}T)$ and $(TR)^{r+1}T(A,i_0) = TR(A_r,-i_r) = T(A_r,i_{r+1}) = (T_{i_{r+1}}A_r,-i_{r+1}) = (A_{r+1},-i_{r+1})$. In the former case, it suffices to show that $A_r \cap W$ is empty. Suppose that it is not empty, and contains, say, w.

Then there exist j, k, v, j' such that $v \in V$, $(w,j) \in \mathscr{D}((TR)^k T)$, and $(TR)^k T(w,j) = (v,j')$, which implies that $w \in J_j$ and that $(w,j) \notin \mathscr{D}((TR)^{k+1}T)$. On the other hand, by Lemma 3d, $(w, -i_r) \in \mathscr{D}((TR)^r T)$ and $(TR)^r T(w, -i_r) \in (A, i_0)$, which implies that $(w, -i_r) \notin \mathscr{D}((TR^{r+1}T))$. Since $w \in J_{-i_r} \cap K - V$, $j = -i_r$. Consequently (since $\mathscr{D}((TR)^k T)$ is decreasing in k), $k = r$, which implies that $(v,j') \in (A, i_0)$. Since this contradicts $A \cap V = \Phi$, the immediate goal has been achieved.

The lemma will be proved once it is shown that (2) cannot hold for all r, for then $k + 1$ can be taken as the first r for which (2) does not hold. If (2) holds for all r, then $A_r \subset J_{-i_r}$, and A_r is congruent to A for all $r \geqq 0$; and, by Lemma 4, the sequence $(A_0, -i_0), (A_1, -i_1), \cdots$, is disjoint. There then is a strictly increasing sequence $r_1, r_2, \cdots$ with $-i_{r_j} = i$ for some i and all j, $A_{r_j} \subset J_i$ for $j = 1,2,\cdots$, and the A_{r_j}'s are disjoint and congruent. But this is impossible.

LEMMA 8. *For every* $A \in \mathscr{A}$, $p(A) \in \mathscr{A}$.

Proof. Since $p^2 = 1_{\mathscr{D}(p)}$, obviously $\mathscr{R}(p) = \mathscr{D}(p)$ and p is $1-1$. By Lemma 7, $\mathscr{D}(p) = K - (V \cup W)$, and, for every $A \in \mathscr{A}$, $p(A)$ is congruent to A and $p(A) \subset J_i$ for some i, so that $p(A)$ is an open arc. Thus the decomposition $K - (V \cup W) = p(K - (V \cup W)) = \bigcup_{A \in \mathscr{A}} p(A)$ is of the type described in Lemma 6, and the assertion follows.

Lemmas 7 and 8 show that *p is a scissor congruence of K onto itself.* They also show that the rigid notions comprising this scissor congruence are in the group generated by $T_1, \cdots, T_n$.

The scissor congruence p decomposes into components in a natural way:

DEFINITION. K_{+-} is the set-theoretic union of all $A \in \mathscr{A}$ such that $A \subset K_+$ and $p(A) \subset K_-$; similarly for K_{-+}, K_{++}, and K_{--}.

Plainly, $K_+ - (V \cup W) = K_{+-} \cup K_{++}$; and $K_- - (V \cup W) = K_{-+} \cup K_{--}$. Moreover, since $p(A) \in \mathscr{A}$ and $p^2(A) = A$ for all $A \in \mathscr{A}$,

(a) $p(K_{+-}) = K_{-+}$,

(b) $p(K_{++}) = K_{++}$,

and

(c) $p(K_{--}) = K_{--}$.

DEFINITION. $p_{+-} = p$ restricted to K_{+-}.

Our next goal is to present a simple intuitive property of the scissor congruence p_{+-} in Lemma 12. But rigor seems to demand two definitions as well as preliminary Lemmas 9, 10, and 11.

DEFINITIONS. Let D and D' be topological discs, let their respective boundaries be J and J', and let A be an open arc with $A \subset J \cap J'$. Say that D and D' are *on the same side of* A if, for every $x \in A$,

(3) there exists $N(x)$ such that $N(x) - A \subset (\operatorname{int} D \cap \operatorname{int} D') \cup (\operatorname{ext} D \cap \operatorname{ext} D')$.

Say that D and D' are *on opposite sides of* A if, for every $x \in A$,

(4) there exists $N(x)$ such that $N(x) - A \subset (\operatorname{int} D \cap \operatorname{ext} D') \cup (\operatorname{ext} D \cap \operatorname{int} D')$.

The proofs of the next three lemmas are not difficult.

LEMMA 9. *D and D' are either on the same or on opposite sides of A.*

LEMMA 10. *If $\operatorname{int} D \cap \operatorname{int} D'$ is empty, then D and D' are on opposite sides of A. If $D \subset D'$, then D and D' are on the same side of A.*

LEMMA 11. *If D and D' are on the same side of A, and D and D'' are on opposite sides of A, then D' and D'' are on opposite sides of A. If D and D' are on opposite sides of A, and D and D'' are on opposite sides of A, then D' and D'' are on the same side of A.*

DEFINITION. For each $A \in \mathscr{A}$, let T_A denote the rigid motion $T_{i_k} \cdots T_{i_0}$ described in Lemma 7. Of course, $T_A(A) = p(A)$.

LEMMA 12. *The scissor congruence p_{+-} preserves sides of arcs. That is, if*
(a) $A \in \mathscr{A}$,
(b) $A \subset J_i$ for some $i > 0$,
(c) $p(A) \subset J_j$ for some $j < 0$,
then $T_A(D_i)$ is on the same side of $p(A)$ as is D_j.

The scissor congruence $K_{++} \to K_{++}$ reverses sides of arcs, as does the scissor congruence $K_{--} \to K_{--}$.

Proof. Let $A \in \mathscr{A}$, and let $i_0, \cdots, i_k$ be as in Lemma 7. Since $-i_r \neq i_{r+1}$, clearly $\operatorname{int} D_{-i_r} \cap \operatorname{int} D_{i_{r+1}} = \Phi$, so by Lemma 10, D_{-i_r} and $D_{i_{r+1}}$ are on opposite sides of A_r for $0 \leqq r \leqq k$. Let $D_{i_0}^r = T_{i_r} \cdots T_{i_0} D_{i_0}$ for $0 \leqq r \leqq k$. As will now be shown, $D_{i_0}^r$ and D_{-i_r} are on the same side of A_r for r even and on opposite sides for r odd. By definition $D_{i_0}^0 = T_{i_0}(D_{i_0}) = D_{-i_0}$; therefore $D_{i_0}^0$ and D_{-i_0} are on the same side of A_0. Now use Lemma 11 repeatedly to see that, for $0 \leqq m \leqq k/2$,

$D_{i_0}^{2m}$ and $D_{-i_{2m}}$ are on the same side of $A_{2m} \Rightarrow$

$D_{i_0}^{2m}$ and $D_{i_{2m+1}}$ are on opposite sides of $A_{2m} \Rightarrow$

$D_{i_0}^{2m+1}$ and $D_{-i_{2m+1}}$ are on opposite sides of $A_{2m+1} \Rightarrow$

$D_{i_0}^{2m+1}$ and $D_{i_{2m+2}}$ are on the same side of $A_{2m+1} \Rightarrow$

$D_{i_0}^{2m+2}$ and $D_{-i_{2m+2}}$ are on the same side of A_{2m+2}.

Since k is even if $A \subset K_{+-}$ and is odd if $A \subset K_{++}$ or if $A \subset K_{--}$, the Lemma follows.

It is now easy to obtain various results of intuitive interest by specializing E_+ and E_-. For example, Lemma 14 is immediate after this preliminary.

LEMMA 13. *If E_+ is a finite union of disjoint strictly convex bodies, (each of which is necessarily a union of some of the D_i) then K_{++} is empty.*

Proof. Suppose $K_{++} \neq \Phi$ and let $A \subset K_{++}$. In the notation of the above proof, $D^k_{i_0}$ and D_{-i_k} are on opposite sides of $p(A)$. Since D_{i_0} and D_{-i_k} are connected, there exist D and D' in the assumed decomposition of E_+ such that $D_{i_0} \subset D$ and $D_{-i_k} \subset D'$. Plainly $A \subset \mathrm{bd}\, D$ because $A = A \cap \mathrm{bd}\, E_+ = A \cap (\mathrm{bd}\, D \cup \mathrm{bd}(E_+ - D))$ $= A \cap \mathrm{bd}\, D$; similarly, $p(A) \subset \mathrm{bd}\, D'$. Let $D^k = T_{i_k} \cdots T_{i_0} D$. By Lemma 10, D^k and D' are on opposite sides of $p(A)$. Let $x \in p(A)$. For any $N(x)$, there is a $y \in (N(x) - \{x\}) \cap p(A)$. Since D^k and D' are strictly convex, $\frac{1}{2}x + \frac{1}{2}y \in$ $\mathrm{int}\, D^k \cap \mathrm{int}\, D'$ which is in contradiction with D^k and D' being on opposite sides of $p(A)$.

LEMMA 14. *If E_+ and E_- are finite unions of disjoint strictly convex topological discs, then K_+ is scissor-congruent to K_-.*

Proof. Immediate from Lemma 13.

Other conclusions are similarly easy to derive. For example, *the only convex body that is scissor-congruent to a polygon is itself a polygon.* In particular, the circle is not scissor-congruent to the square.

Though Jordan arcs can have positive 2-dimensional Lebesgue measure, it is easy to verify:

LEMMA 15. *If K has two-dimensional Lebesgue measure zero, then E_+ and E_- have the same area.*

LEMMA 16. *Let E_+ and E_- be finite unions of disjoint compact convex planar bodies. If E_+ has the same area as E_- and K_+ is scissor-congruent to K_-, then E_+ is scissor-congruent to E_-.*

Proof. The scissor congruence of K_+ and K_-, implies the existence of convex arcs $A_1, \cdots, A_n, A'_1, \cdots, A'_n$ and rigid motions $M_1, \cdots, M_n$ such that: $K_+ = \cup A_i$; $K_- = \cup A'_i$; $M_i(A_i) = A'_i$; $A_i \cap A_j$ and $A'_i \cap A'_j$ consist of at most one point each for $i \neq j$. Let $P_1, \cdots, P_n$ and $P'_1, \cdots, P'_n$ be the end points of the arcs $A_1, \cdots, A_n$ and $A'_1, \cdots, A'_n$ respectively. Suppose first that E_+ and E_- are themselves convex bodies. The P_i and P'_i are the vertices (not necessarily in order) of convex polygons P and P' inscribed respectively in the convex bodies E_+ and E_- respectively.

Each arc A_i together with the chord joining its end points determines a sector S_i of E_+. Plainly M_i maps S_i onto S'_i. Consequently, S_i and S'_i have the same area as do their unions $\cup S_i$ and $\cup S'_i$. Therefore, $E_+ - \cup S_i$ has the same area as $E_- - \cup S'_i$, that is, the interiors of P and P' have the same area. Since P and P' have the same area, Bolyai's theorem applies to show that P and P' are scissor-congruent. Since a scissor congruence of P and P' and isometries of S_i onto S_i' for all i determines a unique scissor congruence of E_+ with E_-, the proof is complete if E_+ and E_- are convex. The argument is easily modified to handle the general case.

THEOREM 1. *Suppose that E_+ and E_- are finite unions of disjoint compact strictly convex planar bodies. Then E_+ and E_- are scissor-congruent if and only if they have the same area and their boundaries are scissor-congruent.*

Proof. Apply Lemmas 14, 15, and 16.

Since the only convex body whose boundary is scissor-congruent to a circle, is a congruent circle, one gets

COROLLARY. *A circular disc is scissor-congruent to no other strictly convex body.*

For a slight generalization of Theorem 1 and of the italicised statement appearing after Lemma 14, introduce two definitions: an arc is *elementary* if it is either strictly convex or a straight line segment; a convex body is *elementary* if its boundary consists of a finite number of elementary arcs.

PROPOSITION 2. *An elementary convex body E_+ is scissor-congruent to a convex body E_- if and only if E_- is elementary, E_- has the same area as E_+, and the strictly convex portion of the boundary of E_+ is scissor-congruent to that of E_-.*

We are grateful to Glen Bredon for showing us how to remove "strictly" from the Corollary, and to Branko Grünbaum for subsequently pointing out to us that Proposition 2, together with an interesting result of Blaschke [3, Chapter II, §6], implies a considerable further improvement of the Corollary, namely:

THEOREM 2. *If an ellipse E_+ is scissor-congruent to a convex body E_- then there is a rigid motion carrying E_+ onto E_-.*

We do not know whether any convex body other than an ellipse is scissor-congruent to no convex body other than itself. Nor do we know how to formulate and prove a theorem in the spirit of Proposition 1 to the effect that a cube in three dimensions is not scissor-congruent to a ball.

REFERENCES

1. Boltyanskii, V.G., 1963, Equivalent and Equidecomposable Figures, D.C. Heath and Co., Boston.

2. Enriques, Federigo, 1912, *Questioni Riguardanti le Mathematiche Elementari*, Volume 1, Articoli di U. Amaldi, "Sulla teoria della equivalenza", Bologna, Nicola Zanichelli, pp. 145–198.

3. Fejes Tóth, L., 1953, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer.

4. Hadwiger, H., 1957, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer.

5. Sierpinski, Waclaw, 1954, On the Congruence of Sets and Their Equivalence by Finite Decomposition, Lucknow University.

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